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The XXZ model of arbitrary spin by direct solution of the Baxter equation

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Abstract
The method of separation of variables (SoV) is employed for the spectral problem of the XXZ model. The Baxter difference equation is resolved by means of a special isotropic asymptotic expansion. States are identified by multiplicities of limiting values of the Bethe parameters. As an application, the statistical properties of integral spectra are investigated. It is shown that the power function gives the more correct description of nearest-neighbour spacing distribution density at intermediate spacings as compared with the exponential.

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1. Introduction

The XXZ model is an integrable periodic chain of N particles where between adjacent elements a spin interaction exists. Let s be the spin of each particle; the Hamiltonian of the model reads

$$H = \sum_{n=1}^{N} J_s\left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+, S_n^3, S_{n+1}^3\right), \quad \mathbf{S}_{N+1} \equiv \mathbf{S}_1. \tag{1}$$

Here J_s is a certain polynomial [1] of order $4s$ on the spin component operators $\mathbf{S}_n = \left\{S_n^3, S_n^+, S_n^-\right\}$, which obey the quantum Lie algebra $U_q(sl(2))$ commutation relations

$$\left[S_n^3, S_n^\pm\right] = \pm S_n^\pm, \qquad \left[S_n^+, S_n^-\right] = \frac{\sin(2\varkappa S_n^3)}{\sin\varkappa}, \tag{2}$$

$$\cos\left(\varkappa\left(2S_n^3+1\right)\right) - 2S_n^- S_n^+ \sin^2\varkappa = \cos(\varkappa(2s+1)). \tag{3}$$

The special form of dependence J_s on $S^\pm$ means a partial isotrophy of the system; a real parameter $\varkappa$ determines the extent of anisotrophy. In the limiting case $\varkappa \to 0$ algebra

$U_q(sl(2))$ turns into standard $sl(2)$ and a completely isotropic XXX model appears. At $s = 1/2$ XXZ is a well-known partially isotropic chain introduced by Heisenberg [2]:

$$H = \sum_{n=1}^{N} \frac{1}{2} \left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ \right) + \cos \varkappa S_n^3 S_{n+1}^3. \tag{4}$$

The integrability of the model can be described by means of the R-matrix formalism of the quantum inverse scattering method [3]. The local Lax matrix for the XXZ

$$L_n(u) = \begin{pmatrix} -\mathrm{i} \sinh \left(u - c_n + \mathrm{i}\varkappa S_n^3 \right) & S_n^- \sin \varkappa \\ S_n^+ \sin \varkappa & -\mathrm{i} \sinh \left(u - c_n - \mathrm{i}\varkappa S_n^3 \right) \end{pmatrix} \tag{5}$$

obeys the fundamental commutation relations with the trigonometric R-matrix. The monodromy matrix

$$T(u) \equiv \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \stackrel{\text{def}}{=} L_N(u) \dots L_1(u) \tag{6}$$

also satisfies the same relations. Its trace $t(u) = A(u) + D(u)$ serves as a generating function for the integrals of motion

$$t(u) = I_1^+ \mathrm{e}^{Nu} + I_2^+ \mathrm{e}^{(N-2)u} + \dots + I_2^- \mathrm{e}^{-(N-2)u} + I_1^- \mathrm{e}^{-Nu}. \tag{7}$$

The leading coefficients of $t(u)$ coincide up to the complex factor $I_1 \stackrel{\text{def}}{=} (\mathrm{i})^N \mathrm{e}^{\sum c_n} I_1^+ = (-\mathrm{i})^N \mathrm{e}^{-\sum c_n} I_1^-$. So exactly N independent coefficients form the basis of the ring of mutually commuting conserved quantities. Hamiltonian (1) belongs to this ring [1].

The spectral problem is formulated as

$$t(u)\Psi = \tau(u)\Psi, \tag{8}$$

where eigenfunction Ψ does not depend on u and the coefficients of the exponential polynomial $\tau(u)$ are target values of the integrals of motion.

The usual way to solve (8) is via an algebraic Bethe ansatz. According to this method the eigenfunction is constructed as

$$\Psi = B(u_M) B(u_{M-1}) \dots B(u_1) \Psi_0, \tag{9}$$

where Ψ_0 is a ferromagnetic vacuum state such that $C(u)\Psi_0 = 0$ for any u and $u_1, u_2, \dots, u_M$ is the set of complex parameters called the Bethe vector. The length of the Bethe vector M, meaning the number of excitations, fixes the value of the first integral of motion

$$I_1 = \frac{1}{2^{N-1}} \cos \left(\varkappa \sum_{n=1}^{N} S_n^3 \right) = \frac{1}{2^{N-1}} \cos \left(\varkappa \left(Ns - M \right) \right). \tag{10}$$

The substitution of (9) into (8) results in a system of nonlinear equations on Bethe parameters $u_1, u_2, \dots, u_M$

$$\prod_{j=1}^{M} \frac{\sinh \left(u_j - u_m - \mathrm{i}\varkappa \right)}{\sinh \left(u_j - u_m + \mathrm{i}\varkappa \right)} = - \prod_{n=1}^{N} \frac{\sinh \left(u_m - c_n + \mathrm{i}\varkappa s \right)}{\sinh \left(u_m - c_n - \mathrm{i}\varkappa s \right)}, \qquad m = 1, \dots, M. \tag{11}$$

The values of integrals of motion arise from

$$\tau(u) = (-\mathrm{i})^N \prod_{n=1}^{N} \sinh(u - c_n + \mathrm{i}\varkappa s) \prod_{m=1}^{M} \frac{\sinh(u - u_m - \mathrm{i}\varkappa)}{\sinh(u - u_m)} + (-\mathrm{i})^N \prod_{n=1}^{N} \sinh(u - c_n - \mathrm{i}\varkappa s) \prod_{m=1}^{M} \frac{\sinh(u - u_m + \mathrm{i}\varkappa)}{\sinh(u - u_m)}, \tag{12}$$

which is actually an exponential polynomial because the poles at points $u_1, \ldots, u_M$ are cancelled due to (11).

The complexity of Bethe equations (11) makes them inconvenient in practical computations. Moreover, the variety of completely integrable models cannot be treated with the Bethe ansatz because of the absence of the vacuum state. This forces us to look for another approach to solve (8).

2. Separation of variables (SoV)

In the 1990s a new method to solve spectral problem (8) was developed [4–6]. It implies conversion to a representation where the eigenfunction separates into one-variable factors. The eigenfunction in initial representation $\mathbf{x}$ is constructed as

$$\Psi(\mathbf{x}) = \sum_{\substack{\cdots \\ c_1 - \mathrm{i}\varkappa(s-1) \\ v_1 = c_1 - \mathrm{i}\varkappa s}} \cdots \sum_{\substack{\cdots \\ c_{N-1} - \mathrm{i}\varkappa(s-1) \\ v_{N-1} = c_{N-1} - \mathrm{i}\varkappa s}} K(\mathbf{x}\,|v_1, \ldots, v_{N-1})\Phi(v_1, \ldots, v_{N-1}), \tag{13}$$

where a summation over a hyperoctant grid of spacing $\mathrm{i}\varkappa$ in C^{N-1} is carried out. To factorize the eigenfunction in the new representation $\Phi(v_1, \ldots, v_{N-1})$, the special condition is imposed on the transformation kernel

$$C(v_j)K(\mathbf{x}\,|v_1, \ldots, v_{N-1}) = 0, \qquad j = 1, \ldots, N-1. \tag{14}$$

It guarantees A and D act as shift operators on $K(\mathbf{x}\,|v_1, \ldots, v_{N-1})$:

$$A(v_k)K(\mathbf{x}\,|\ldots, v_k, \ldots) = (-\mathrm{i})^N \prod_{n=1}^{N} \sinh(v_k - c_n - \mathrm{i}\varkappa(s+1))K(\mathbf{x}\,|\ldots, v_k - \mathrm{i}\varkappa, \ldots),$$

$$D(v_k)K(\mathbf{x}\,|\ldots, v_k, \ldots) = (-\mathrm{i})^N \prod_{n=1}^{N} \sinh(v_k - c_n + \mathrm{i}\varkappa(s+1))K(\mathbf{x}\,|\ldots, v_k + \mathrm{i}\varkappa, \ldots).$$

As a result of the substitution of (13) into (8) we get

$$\Phi(v_1, \ldots, v_{N-1}) = \prod_{i \neq j} \sinh(v_i - v_j) \prod_{k=1}^{N-1} \varphi_k(v_k), \tag{15}$$

where each factor φ_k obeys the same Baxter equation

$$\tau(v)\varphi(v) = (-\mathrm{i})^N \prod_{n=1}^{N} \sinh(v - c_n - \mathrm{i}\varkappa s)\varphi(v + \mathrm{i}\varkappa) + (-\mathrm{i})^N \prod_{n=1}^{N} \sinh(v - c_n + \mathrm{i}\varkappa s)\varphi(v - \mathrm{i}\varkappa). \tag{16}$$

Thus the separation of variables adds up to a one-dimensional difference equation. Unfortunately, the method gives no direct instructions for resolving (16). Nevertheless, the connection with the Bethe ansatz allows us to define the generic class of solutions. Substituting $\varphi(v) = \prod_{m=1}^{M} \sinh(v - u_m)$ into (16) one can get the same expression for $\tau(v)$ as (12). This means that the target solution of the Baxter equation is an exponential polynomial of order M with zeros as the Bethe parameters.

The technique employed in the following section to work with the Baxter equation is similar to that used in [7], where the case of entirely isotropic *XXX* model is examined. This technique implies the consideration of the simultaneous limit of two model parameters, which allows us to construct the chain of successive approximations. In the first order a quantization

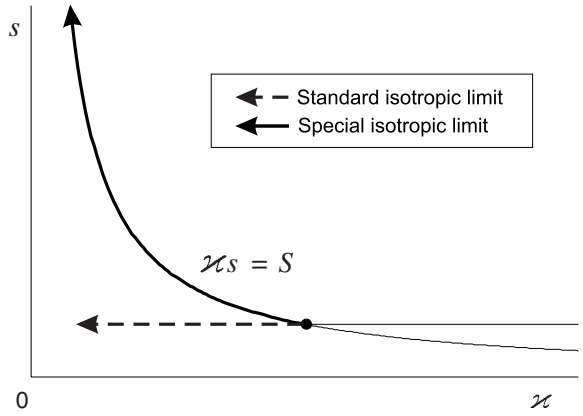

Figure 1. Special isotropic limit.

occurs, so every state is identified by a certain multi-index value. The computations are made in terms of coefficients at poles, originated as the functions are normalized to the first-order approximation ones.

3. Baxter equation in a special isotropic limit

Let us consider the limit $\varkappa \to 0$ constrained by $\varkappa s \stackrel{\text{def}}{=} S \sim \text{const}$ as is shown in figure 1. We substitute asymptotic expansions

$$\tau(v) = \tau_0(v) + \tau_1(v)\varkappa + \tau_2(v)\varkappa^2 + \cdots, \tag{17}$$

$$\varphi(v) = \varphi_0(v) + \varphi_1(v)\varkappa + \varphi_2(v)\varkappa^2 + \cdots \tag{18}$$

into the Baxter equation (16) and simultaneously replace $\varphi(v \pm \mathrm{i}\varkappa)$ by its Taylor series

$$\varphi(v \pm \mathrm{i}\varkappa) = \varphi(v) \pm \mathrm{i}\varphi'(v)\varkappa \mp \frac{\varphi''(v)}{2}\varkappa^2 \pm \cdots. \tag{19}$$

The comparison of coefficients at the same degrees of $\varkappa$ on opposite sides of the equation gives

$$\tau_0(v)\varphi_0(v) = (\Delta^+(v) + \Delta^-(v))\varphi_0(v), \tag{20}$$

$$\tau_1(v)\varphi_0(v) + \tau_0(v)\varphi_1(v) = (\Delta^+(v) + \Delta^-(v))\varphi_1(v) - \mathrm{i}(\Delta^+(v) - \Delta^-(v))\varphi_0'(v), \tag{21}$$

$$\vdots$$

$$\sum_{k=0}^{K} \tau_k(v)\varphi_{K-k}(v) = \sum_{k=0}^{K} \frac{\mathrm{i}^k(\Delta^-(v) + (-1)^k\Delta^+(v))}{k!}\varphi_{K-k}^{(k)}(v) \tag{22}$$

$$\vdots$$

where the symbols

$$\Delta^{\pm}(v) = (-\mathrm{i})^N \prod_{n=1}^{N} \sinh(v - c_n \pm \mathrm{i}S), \tag{23}$$

are introduced. Limiting values of the integrals of motion follow directly from (20)

$$\tau_0(v) = (\Delta^+(v) + \Delta^-(v)). \tag{24}$$

3.1. First approximation

The first approximation allows us to do quantization due to the polynomial nature of $\varphi(v)$ function.

Taking into account (24) we transform formula (21) into

$$\tau_1(v)\varphi_0(v) = -\mathrm{i}(\Delta^+(v) - \Delta^-(v))\varphi_0'(v). \tag{25}$$

This immediately gives

$$\varphi_0(v) = \prod_{n=1}^{N} \sinh^{m_n}(v - r_n), \tag{26}$$

$$\tau_1(v) = 2(-\mathrm{i})^{N-1} \sin NS \prod_{n=1}^{N} \sinh(v - r_n) \sum_{n=1}^{N} m_n \coth(v - r_n), \tag{27}$$

where quantum numbers $\mathbf{m} = \{m_1, m_2, \ldots, m_N\}$ define values of $\tau_1(v)$ at $\varkappa \to 0$ and $r_1, r_2, \ldots, r_N$ are roots of the polynomial $\Delta^+(v) - \Delta^-(v)$:

$$(\Delta^+(v) - \Delta^-(v)) = 2(-\mathrm{i})^N \sin NS \prod_{n=1}^{N} \sinh(v - r_n). \tag{28}$$

It can be shown that $r_1, r_2, \ldots, r_N$ are distinctive at $s > 0$, are real or their imaginary part is multiple of $\frac{\pi}{2}$ and $\sum r_n = \sum c_n - \mathrm{i}\frac{\pi}{2}$. For a homogeneous magnet ($c_1 = c_2 = \cdots = c_N = 0$) they are expressed explicitly as

$$r_n = \operatorname{arctanh}\left(\frac{\tan \varkappa}{\tan \frac{\pi n}{N}}\right), \qquad n = 1, \ldots, N. \tag{29}$$

Roots $r_1, r_2, \ldots, r_N$ are limiting values of the Bethe parameters and the numbers $\{m_1, m_2, \ldots, m_N\}$ are their multiplicities respectively. The sum of the latters is the number M of excitations of the system

$$\sum_{n=1}^{N} m_n = M. \tag{30}$$

Constraint (30) which expresses the value of the first integral of motion (10) is fixed. The multi-index $\mathbf{m}$ length (N) exceeds that in the entirely isotropic case [7] by unity, because due to anisotrophy the square of the total spin of the system is not conserved so the considered quantum space dimension is larger.

3.2. Kth approximation: induction condition

Let us assume that the $(K-1)$th approximation of the scheme is achieved, i.e. functions $\tau_0, \tau_1, \ldots, \tau_{K-1}$ and $\varphi_0, \varphi_1, \ldots, \varphi_{K-2}$ are constructed. The aim is to obtain the next, Kth, correction to the Baxter equation solution φ_{K-1} and simultaneously coefficients at $\varkappa^K$ in the integral decompositions expressed by means of τ_K.

Due to (24), terms at $k = 0$ on either side of (22) cancel and we arrive at the inhomogeneous first-order differential equation for $\varphi_{K-1}(v)$:

$$\mathrm{i}(\Delta^+(v) - \Delta^-(v))\varphi_{K-1}'(v) + \tau_1(v)\varphi_{K-1}(v) = \sum_{k=2}^{K} \left(\frac{\mathrm{i}^k(\Delta^-(v) + (-1)^k\Delta^+(v))}{k!}\varphi_{K-k}^{(k)}(v) - \tau_k(v)\varphi_{K-k}(v)\right). \tag{31}$$

Function $\varphi_0(v)$ (26) is a solution of the corresponding homogeneous equation. Our goal is the partial solution of the inhomogeneous one, namely

$$\varphi_{K-1}(v) = \frac{\prod_{n=1}^{N}\sinh^{m_n}(v-r_n)}{2(-\mathrm{i})^{N-1}\sin NS} \times \int\sum_{k=2}^{K}\left(-\frac{\mathrm{i}^k(\Delta^-(v)+(-1)^k\Delta^+(v))}{k!}\varphi_{K-k}^{(k)}(v)+\tau_k(v)\varphi_{K-k}(v)\right) \times\frac{\mathrm{d}v}{\prod_{n=1}^{N}\sinh^{m_n+1}(v-r_n)}. \tag{32}$$

In order for φ_{K-1} to be an exponential polynomial, the integrand in (32) must have no poles of first order. This condition defines coefficients of $\tau_K(v)$.

Let us introduce new quantities

$$\widetilde{\tau}_k(v) = \frac{\tau_k(v)}{2(-\mathrm{i})^{N-1}\sin NS\prod_{n=1}^{N}\sinh(v-r_n)} \tag{33}$$

and

$$\widetilde{\varphi}_k(v) = \frac{\varphi_k(v)}{\prod_{n=1}^{N}\sinh^{m_n}(v-r_n)}. \tag{34}$$

Corrections to integrals of motion are described entirely by N coefficients at first-order poles of $\widetilde{\tau}_k$ at points $r_1, r_2, \ldots, r_N$. The sum of the coefficients is constrained by the kth correction to the first integral of motion (10) in the considered limit

$$I_1 = \cos(NS)+\sin(NS)M\varkappa-\frac{\cos(NS)M^2}{2!}\varkappa^2-\frac{\sin(NS)M^3}{3!}\varkappa^3+\cdots.$$

Corrections to the Baxter equation solution are uniquely defined by coefficients at $\widetilde{\varphi}_k$ poles of order up to m_n inclusive, at points r_n, $n=1,\ldots,N$. The total number of coefficients is M (30).

In terms of pole characteristics the formula (32) reads

$$\widetilde{\varphi}_{K-1} = \int\left(\left\{-\sum_{k=2}^{K}((\widetilde{\tau}_0-\mathrm{i})+(-1)^k(\widetilde{\tau}_0+\mathrm{i}))\frac{\mathrm{i}^k\left(\widetilde{\varphi}_{K-k}\prod_{n=1}^{N}\sinh^{m_n}(v-r_n)\right)^{(k)}}{2k!\prod_{n=1}^{N}\sinh^{m_n}(v-r_n)} +\sum_{k=2}^{K-1}\widetilde{\tau}_k\widetilde{\varphi}_{K-k}\right\}+\widetilde{\tau}_K\right)\mathrm{d}v. \tag{35}$$

The expression in braces has poles of order from the first to the (m_n+1)th inclusive, at points r_n, $n=1,\ldots,N$. Pole coefficients of $\widetilde{\tau}_K$ are equated to coefficients at first-order poles. As a result, the unwanted logarithmic component cancels and the function $\widetilde{\varphi}_{K-1}$ has the required pole characteristics, i.e. its poles order does not exceed m_n.

Thus the solution of the Baxter equation (16) reduces to the following procedure. The first step is to determine roots $r_1,\ldots,r_N$ of the exponential polynomial $\Delta^+(v)-\Delta^-(v)$ (28) and assign the value of multi-index $\mathbf{m}=\{m_1,\ldots,m_N\}$. This allows us to initiate an asymptotic approximation chain (35) starting from $\widetilde{\varphi}_0(v)\equiv 1$ and

$$\widetilde{\tau}_0(v) = -\mathrm{i}\frac{\Delta^+(v)+\Delta^-(v)}{\Delta^+(v)-\Delta^-(v)}, \qquad \widetilde{\tau}_1(v) = \sum_{n=1}^{N}m_n\frac{\cosh(v-r_n)}{\sinh(v-r_n)}.$$

The computation of integral (35) adds up to a manipulation with the finite sets of real numbers. These numbers are actually the coefficients at poles of various orders at points $r_1,\ldots,r_N$

Table 1. Normalized energy values E/N for states $\{0, 1, 1, 0\}$, $\{1, 2, 0, 0\}$ and $\{2, 0, 0, 1\}$ depending on the approximation order K. $N = 4, c_1 = c_2 = c_3 = c_4 = 0, s = 1, \varkappa = \pi/10$. In the last row the values reported in [8] are placed.

	E/N		
K	$\mathbf{m} = \{0, 1, 1, 0\}$	$\mathbf{m} = \{1, 2, 0, 0\}$	$\mathbf{m} = \{2, 0, 0, 1\}$
3	−0.434 413 9992	−0.199 671 5487	−0.073 730 7436
5	−0.443 627 1461	−0.202 214 7410	−0.075 331 1657
7	−0.444 111 7378	−0.202 309 1353	−0.075 548 0019
9	−0.444 122 0336	−0.202 269 9562	−0.075 593 0131
11	−0.444 120 4735	−0.202 256 9562	−0.075 604 2315
13	−0.444 120 2121	−0.202 254 5161	−0.075 607 0699
15	−0.444 120 1897	−0.202 254 2334	−0.075 607 7766
17	−0.444 120 1888	−0.202 254 2337	−0.075 607 9515
19	−0.444 120 1889	−0.202 254 2444	−0.075 607 9948
21	−0.444 120 1889	−0.202 254 2478	−0.075 608 0056
23	−0.444 120 1889	−0.202 254 2485	−0.075 608 0083
25	−0.444 120 1889	−0.202 254 2486	−0.075 608 0090
Alcaraz *et al* [8]	−0.444 1201	−0.202 2542	−0.075 6080

of considered functions. At every step the correction to the normalized value (33) of the monodromy matrix trace (7) is evaluated: $\widetilde{\tau}_2, \widetilde{\tau}_3, \widetilde{\tau}_4$, etc., which gives the corresponding corrections to the integrals of motion.

4. Results of computations

To ensure the correctness of computations we compare their results with the numeric data of [8], where energy levels of the 4-particle homogeneous *XXZ* model were found by the direct solution of the Bethe equations (11). The energy levels were obtained from the Bethe parameters via the following formula:

$$E = \frac{\sin^2(2\varkappa s)}{2s} \sum_{m=1}^{M} \frac{1}{\cos(2\varkappa s) - \cosh(2u_m)}.$$

In spite of the completely different algorithms, the coincidence up to seven digits presented in [8] is achieved. The discrepancy in the last digit for the state with multi-index $\{0, 1, 1, 0\}$ registered in table 1 is apparently due to incorrect rounding in [8].

4.1. XXZ nearest-neighbour spacing distribution

The investigation of statistical properties of quantum-mechanical system spectra, in particular, nearest-neighbour spacing distribution of levels is of great practical importance. The theoretical treatment of this issue was initiated by the work of Wigner [9] where the following formula for level nearest-neighbour spacing distribution density had been proposed:

$$P(\sigma) = 2\beta\sigma \exp(-\beta\sigma^2). \tag{36}$$

Here coefficient β is defined via the average level spacing. Wigner's formula was later specified and extended over the variety of systems with different symmetries within the framework of random matrix theory [10]. The integrable systems stand out against the others because of

the absence of level repulsion; in other words, their distribution density $P(\sigma)$ does not tend to zero at $\sigma \to 0$ unlike (36). It is asserted [11] that the level spacings for the integrable systems are distributed according to the damped exponential law

$$P(\sigma) = \beta \exp(-\beta\sigma). \tag{37}$$

Formula (37) was obtained in the semi-classical limit and so should be considered only as an approximation.

In this connection, it is of interest to estimate by direct computations the extent of applicability of (37) to the *XXZ* model. On the other hand, this enables us to check the effectiveness of the technique stated in the previous section in practice.

Calculations are made for the chain of $N = 100$ particles with $M = 3$ excitations, which gives $\binom{N+M-1}{M} = 171\,700$ different states. The spin of each particle is $s = 3/2$, $\varkappa = 0.25$ and shift parameters c_n, $n = 1, \ldots, N$, are randomly distributed within the interval $[-5 \ldots 5]$.

For each state an asymptotic approximation as described at the ending of the previous section was constructed. The series was truncated as the 10-digit accuracy was achieved or the asymptotic started to diverge. In the latter condition, which occurs for about 17% of states, the Bethe equations were applied to define more precisely the obtained values. In any case the length of the series used varied from 5 to 30.

In figure 2, the spectral characteristics for several integrals of various orders (I_{13}^-, I_{26}^+, I_{38}^-, I_{51}) are shown. Integral values are normalized to symmetric functions on shifts c_n, $n = 1, \ldots, N$:

$$\widetilde{I}_j^{\pm} = \frac{I_j^{\pm}}{2^{-N}\,\mathrm{e}^{\mp(c_1+\cdots+c_N)} \sum_{n_1<n_2<\cdots<n_{j-1}} \exp\left(\pm 2(c_{n_1}+c_{n_2}+\cdots+c_{n_{j-1}})\right)},$$
$$j = 2, \ldots, [N/2]+1. \tag{38}$$

Histograms of 500 bars on the left-hand side of figure 2 represent the densities of level distribution p_j; ones on the right-hand side show the densities of the corresponding nearest-neighbour spacing distribution P_j. One can see the characteristics of spectrum depend on the integral order j.

The integrals of low order have the highly clusterized spectrum. It consists of strongly isolated domains of eigenvalue concentration. Each large-scale cluster contains ~5000 values and is characterized by the unit value of the specific index m_n, which corresponds to parameter r_n (26) with a large negative real part for 'positive' integrals and a large positive real part for the 'negative' ones. e.g. a cluster which can be seen by the unaided eye in figure 2 at $\widetilde{I}_{13}^- \approx -3.25$ is formed by the eigenvalues with multi-indices like $\mathbf{m} = \{\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, 1, \ldots\}$ (unit stands in the 10th vacancy, $r_{10} \approx 3.39 + \frac{\pi}{2}\mathrm{i}$). Large-scale cluster consists of a hundred of twofold-overlapped small clusters of 100 eigenvalues. The fine structure of the small clusters is determined by the relations of r_n with large positive real part for 'positive' integrals and large negative real part for the 'negative' ones. The strongly pronounced clusterization is caused primarily by an inhomogeneity due to which the eigenvalues are expressed via the parameters of greatly unequal values. In a homogeneous case the clusterization also exists but in a lesser extent.

As the order of integral grows, clusters become wider and wider with respect to the distance between them and mutually overlap. Finally, most of eigenvalues of higher order integrals form a common single bulk with the chaotic inner structure (I_{38}^-, I_{51} spectra in figure 2).

On the nearest-neighbour spacing distribution of low order integral the distinct peaks are observed. This fact is in correlation with the observed spectrum clusterization for such an integral. Indeed, the fine structure of the small clusters considerably coincides because it is

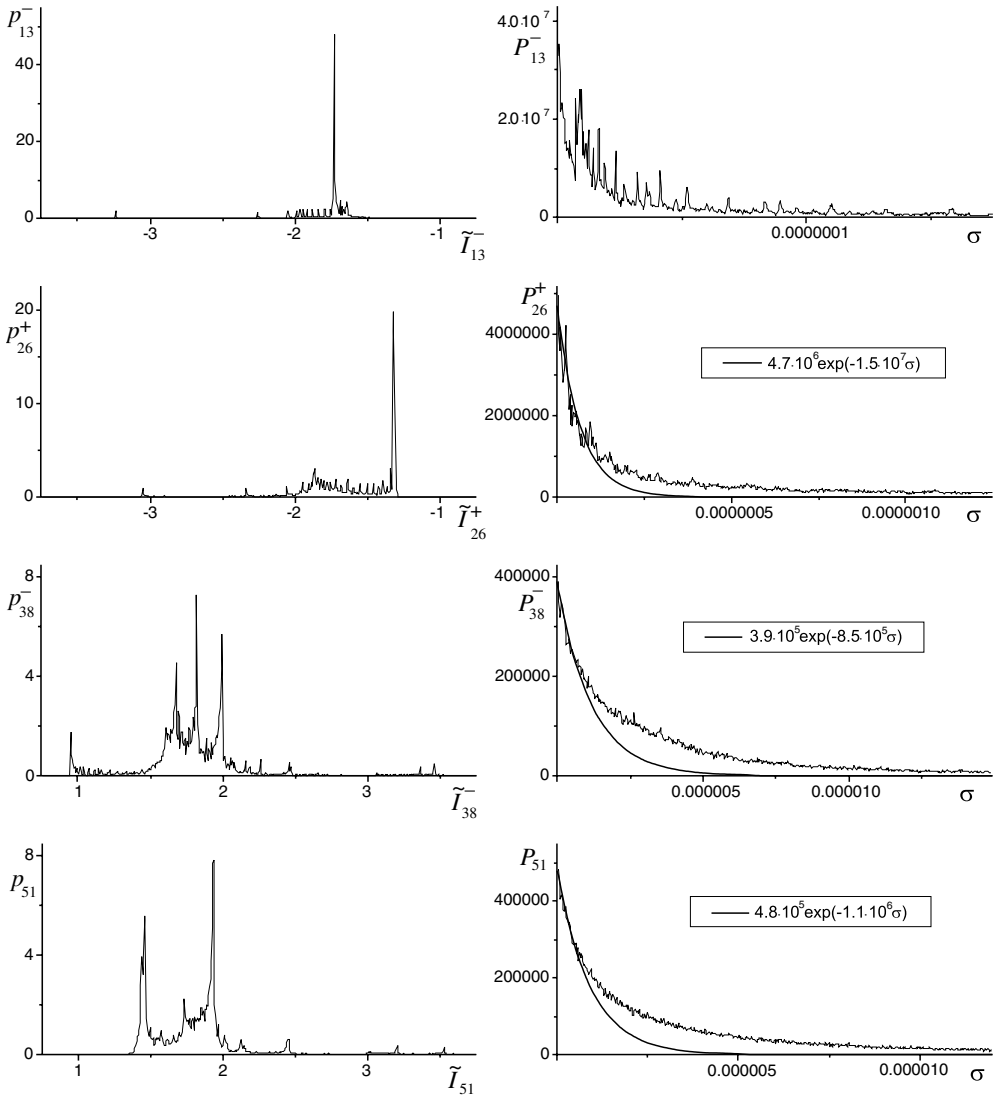

Figure 2. Eigenvalue distributions p_j (left) and nearest-neighbour spacing distributions P_j (right).

determined by the same set of parameters; as a result, some spacings are found much more frequently than the others. It looks realizable to solve an inverse problem, i.e. to obtain the values of r_n using the peak locations on P_j and thus to get information about inhomogeneity parameters c_n, $n = 1, \ldots, N$.

Unless deflections concerned with the clusterization are taken into account, the description of the spacing distribution by formula (37) is rather good at small σ (see smooth curves on the right-hand side graphics in figure 2). But at σ comparable with the average distance between the levels β and the larger ones this description becomes evidently incorrect, since the distribution density decreases considerably slower as compared with the exponential. The more correct approximation at such σ is provided by the power function $P_j \sim \sigma^{-a_j}$, which

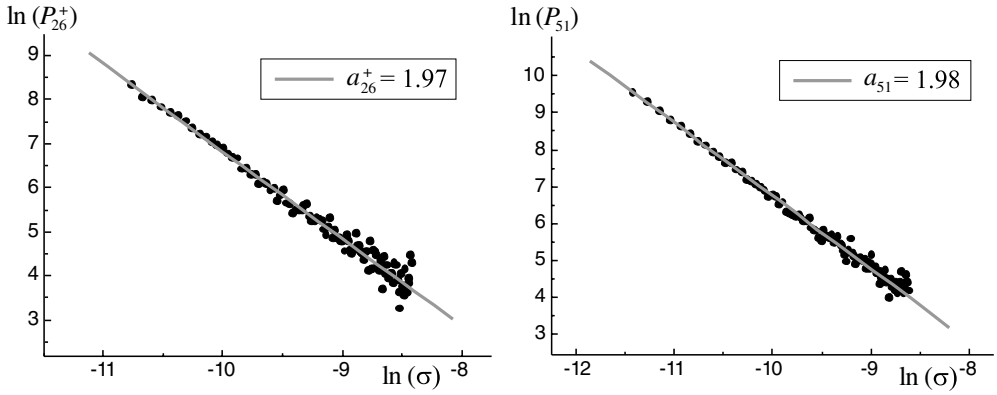

Figure 3. Nearest-neighbour spacing distribution at large σ for I_{26}^{+} (left) and I_{51} (right).

Table 2. Superscript a_j of power function, approximating nearest-neighbour spacing distribution density, for several integrals of motion.

	I_{26}^{+}	I_{28}^{-}	I_{31}^{+}	I_{33}^{-}	I_{36}^{+}	I_{38}^{-}	I_{41}^{+}	I_{43}^{-}	I_{46}^{+}	I_{48}^{-}	I_{51}
a_j	1.97	2.10	2.03	1.96	2.11	2.03	2.18	1.90	1.97	2.12	1.98

is displayed in figure 3. The magnitude of superscript a_j varies within the rather close limits namely $\sim 2 \pm \frac{1}{4}$, as is shown in table 2.

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